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Analysis of the Finite Cone-Plate Viscometer and of the Finite Parallel-Plate Viscometer

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Synopsis

The analyses for flow of a simple fluid of Noll both in the infinite cone-plate viscometer and in the infinite parallel-plate viscometer are shown to be consistent with the existence of a free surface in the finite instruments. Neglected in the description of the free surface are: the effect of any gradients in surface tension, the effect of the rate-of-deformation of the free surface upon the surface stress tensor, the effect of mass transfer at the free surface, inertial and gravitational effects in the free surface, and distortion of the free surface at solid boundaries.

1. INTRODUCTION

Two instruments which are currently being used in normal-stress measurements are the cone-plate viscometer and the parallel-plate viscometer.[1,2] In the analyses of the experiments[3-5] the assumption is made that the radial dimensions of the instruments are without limit, i.e., no attempt is made to account for edge effects. In what follows we ask whether the analyses of the infinite cone-plate viscometer and of the infinite parallel-plate viscometer are consistent with the existence of a free surface in the finite instruments.

Noll's theory of simple fluids[6-8] is assumed to describe the behavior of the fluids being tested. In treating behavior at the free surface, we neglect the effect of any gradients in surface tension, the effect of the rate-of-deformation of the free surface upon the surface stress tensor, the effect of mass transfer at the surface, inertial and gravitational effects in the free surface.[9,10]

2. FINITE CONE-PLATE VISCOMETER

We begin by examining the infinite cone-plate viscometer for a simple fluid of Noll.[6-8] Following this we formulate the boundary conditions at the free surface and ask whether these boundary conditions are consistent with the analysis for the infinite geometry.

2.1. Infinite Cone-Plate Viscometer

This analysis follows directly from the general discussion of Coleman and Noll for the class of viscometric flows of simple fluids[7,8] and parallels their discussion for other geometries.

We choose a spherical coordinate system with origin at the apex of the cone and θ the angle measured between the axis of the cone and any position vector. The z component of torque required to drive the cone at a constant angular velocity Ω is $-T_z$; the plate is fixed in space. The angle between the cone and the plate is $(\pi/2 - \Theta)$.

We assume that there is only one nonzero component of velocity,* v_φ, and that $(v_\varphi/r \sin\theta)$ is a function of θ alone. Under these conditions Coleman and Noll show that there are four nonzero components of stress $(t_{\varphi\theta}, t_{\theta\theta}, t_{rr}, t_{\varphi\varphi})$ and that the behavior of a simple fluid in this flow (since it is one of the class of viscometric flows) is described by three material functions

$$\tau(\varkappa) = t_{\varphi\theta} \tag{1}$$

$$\sigma_1(\varkappa) = t_{\theta\theta} - t_{rr} \tag{2}$$

$$\sigma_2(\varkappa) = t_{\varphi\varphi} - t_{rr} \tag{3}$$

where

$$\varkappa = \sin\theta \, d(v_\varphi r \sin\theta)/d\theta \tag{4}$$

Further we have that $\tau(\varkappa)$ is an odd function of $\varkappa$ and that $\sigma_1(\varkappa)$ and $\sigma_2(\varkappa)$ are even functions of $\varkappa$:

$$\begin{aligned} \tau(-\varkappa) &= -\tau(\varkappa) \\ \sigma_1(-\varkappa) &= \sigma_1(\varkappa) \\ \sigma_2(-\varkappa) &= \sigma_2(\varkappa) \end{aligned} \tag{5}$$

Previous discussions[2,3,5] indicate that, if there is to be only one nonzero component of velocity in this flow, inertial effects must be neglected. We shall assume that the external body force vector per unit mass f_i represents the acceleration of gravity and hence may be represented by a potential†

$$f_i = \psi_{,i} \tag{6}$$

If we define

$$\Phi = p + \rho\psi \tag{7}$$

* Components of vectors and tensors with indices corresponding to the three spherical coordinates (r, θ, φ) are physical components.[11a]

† Latin indices indicate tensors with respect to coordinate transformations in 3-space; Greek indices denote tensors with respect to surface coordinate transformations. Comma notation stands for covariant differentiation[11b] and the summation convention is employed throughout. In Cartesian, orthogonal coordinates there is no distinction between the covariant and contravariant components of a tensor, and the comma denotes partial differentiation with respect to the space coordinate whose index follows. For the equations of continuity and of motion in special coordinate systems see, for example, Bird et al.[12]

under the above limitations the stress equations of motion become

$$0 = -\frac{\partial \Phi}{\partial r} - \frac{(\sigma_1 + \sigma_2)}{r} \tag{8}$$

$$0 = -\frac{\partial \Phi}{\partial \theta} + \frac{d\tau_{\theta\theta}}{d\theta} + (\sigma_1 - \sigma_2)\cot\theta \tag{9}$$

$$0 = -\frac{\partial \Phi}{\partial \varphi} + \frac{1}{\sin\theta}\frac{d(\sin^2\theta\ \tau_{\theta\varphi})}{d\theta} \tag{10}$$

Since Φ must be a periodic function of φ, from eq. (10) $\partial\Phi/\partial\varphi = 0$ and

$$\sin^2\theta\ \tau_{\theta\varphi} = C_1 = \text{constant} \tag{11}$$

The rectangular Cartesian components of the torque vector T_m which the fluid exerts on the cone are obtained by integrating over the surface of the cone the rectangular Cartesian components of the vector product of the position vector R^j with the stress vector t^k (which represents the force per unit area that the fluid exerts on the surface of the cone at each point).

$$T_m = \int_S \frac{\partial x^i}{\partial y^m}\ \epsilon_{ijk} R^j t^k dS \tag{12}$$

In terms of the stress tensor t^{km} and the outwardly directed unit normal vector to the surface n_p we have

$$T_m = \int_S \frac{\partial x^i}{\partial y^m}\ \epsilon_{ijk}\ R^j t^{kp} n_p dS \tag{13}$$

By a straightforward computation we find that

$$T_z = \int_0^{2\pi}\int_0^R \tau_{\theta\varphi} r^2 \sin^2\Theta\, dr\, d\varphi \tag{14}$$

$$= 2\pi C_1 R^3/3 \tag{15}$$

From eqs. (1) and (11)

$$\tau_{\theta\varphi} = t_{\theta\varphi} = \frac{3T_z}{2\pi\ R^3 \sin^2\theta} = \tau(\varkappa) \tag{16}$$

If τ is an invertible function[7]

$$\varkappa = \sin\theta \frac{d}{d\theta}\left[\frac{v_\varphi}{r\sin\theta}\right] = \tau^{-1}\left[\frac{3T_z}{2\pi\ R^3\sin^2\theta}\right] \tag{17}$$

which may be integrated to yield

$$\Omega = \int_{\pi/2}^{\Theta} \frac{d}{d\theta}\left[\frac{v_\varphi}{r\sin\theta}\right] d\theta = \int_{\pi/2}^{\Theta} \tau^{-1}\left[\frac{3T_z}{2\pi\ R^3\sin^2\theta}\right]\frac{d\theta}{\sin\theta} \tag{18}$$

This last equation may in turn be differentiated to give

$$\sin\Theta \frac{d\Omega}{d\Theta} = \tau^{-1}\left[\frac{3T_z}{2\pi R^3 \sin^2\Theta}\right] \tag{19}$$

which may be used to generate the function $\tau(\varkappa)$ from experimental data.

Returning to the other two equations of motion, from eqs. (8) and (9) we have

$$\Phi - \Phi\Big|_{\substack{r=R\\ \theta=\pi/2}} = \int_R^r - \frac{(\sigma_1+\sigma_2)}{r}\,dr + \int_{\pi/2}^{\theta}\left[\frac{d\tau_{\theta\theta}}{d\theta} + (\sigma_1-\sigma_2)\cot\theta\right]d\theta \qquad (20)$$

$$\Phi - \Phi\Big|_{\substack{r=R\\ \theta=\pi/2}} = (\sigma_1+\sigma_2)\ln(R/r) + \tau_{\theta\theta} - \tau_{\theta\theta}\Big|_{\theta=\pi/2} + \int_{\pi/2}^{\theta}(\sigma_1-\sigma_2)\cot\theta\,d\theta \qquad (21)$$

Upon rearrangement of eq. (21) using eq. (7) we obtain

$$t_{\theta\theta} = t_{\theta\theta}\Big|_{\substack{r=R\\ \theta=\pi/2}} + \rho\psi - \rho\psi\Big|_{\substack{r=R\\ \theta=\pi/2}} - (\sigma_1+\sigma_2)\ln(R/r) - \int_{\pi/2}^{\theta}(\sigma_1-\sigma_2)\cot\theta\,d\theta \qquad (22)$$

2.2 Boundary Conditions at the Free Surface in the Finite Cone-Plate Viscometer

Under the restrictions stated in the introduction, the surface equations of motion[9,10] reduce to

$$2Hn^i\,\sigma = -t^{ij}_{(1)}\,n_{(1)j} - t^{ij}_{(2)}\,n_{(2)j} \qquad (23)$$

Here $n^j_{(K)}$ is the unit vector normal to the surface and outwardly directed into phase K, $t^{ij}_{(K)}$ is the stress tensor adjacent to the surface in phase K, σ is surface tension, n^i is a unit vector normal to the surface such that $(x^i_{;1}, x^i_{;2}, n^i)$ have the same orientation as the tangents to the spatial coordinate curves,[11b] given by

$$n_i = \frac{1}{2}\,\epsilon^{\alpha\beta}\epsilon_{ijk}\,x^j_{;\alpha}\,x^k_{;\beta} \qquad (24)$$

$$x^i_{;\alpha} = \partial x^i/\partial u^\alpha \qquad (25)$$

H is the Mean curvature of the surface[11c] and is given by

$$H = \frac{1}{2}\,a^{\alpha\beta}\,b_{\alpha\beta} \qquad (26)$$

where $a_{\alpha\beta}$ is a surface metric tensor[11d]

$$a_{\alpha\beta} = g_{ij}\,x^i_{;\alpha}\,x^j_{;\beta} \qquad (27)$$

and $b_{\alpha\beta}$ is a symmetric tensor in Gauss's formulae[11e]

$$b_{\alpha\beta} = -\,g_{mn}\,n^m_{,\alpha}\,x^n_{;\beta} \qquad (28)$$

We observe that for the case we wish to consider the surface has no normal component of velocity and that the surface is one of revolution. Accordingly, the surface coordinates are chosen to be

$$\begin{aligned} u^1 &= x^1 = \theta \\ u^2 &= x^2 = \varphi \end{aligned} \tag{29}$$

and on the surface

$$x^3 = r = f(u^1) \tag{30}$$

From eq. (27) the components of the surface metric tensor are

$$\begin{aligned} a_{11} &= (f)^2 + (f')^2 \\ a_{12} &= 0 \\ a_{22} &= (f)^2 \sin^2 u^1 \end{aligned} \tag{31}$$

Let us denote the liquid in the viscometer as phase 1 and the surrounding air as phase 2. Then in the coordinate system described above

$$n^i = n^i_{(2)} = -n^i_{(1)} \tag{32}$$

and from eq. (24)

$$\begin{aligned} n_1 &= rn_\theta = \frac{-ff'}{[(f)^2 + (f')^2]^{1/2}} \\ n_2 &= r \sin\theta\, n_\varphi = 0 \\ n_3 &= n_r = \frac{f}{[(f)^2 + (f')^2]^{1/2}} \end{aligned} \tag{33}$$

From eq. (28)

$$\begin{aligned} b_{11} &= \frac{ff'' - (f)^2 - 2(f')^2}{[(f)^2 + (f')^2]^{1/2}} \\ b_{12} &= 0 \\ b_{22} &= \frac{\sin u^1 \cos u^1\, ff' - \sin^2 u^1\,(f)^2}{[(f)^2 + (f')^2]^{1/2}} \end{aligned} \tag{34}$$

and we find the mean curvature of the surface from eq. (26) to be

$$H = \frac{ff'' - 2(f)^2 - 3(f')^2 + ff' \cot u^1 + \cot u^1\,(f')^3/f}{2[(f)^2 + (f')^2]^{3/2}} \tag{35}$$

If we neglect all viscous forces in the surrounding air stream where the pressure is everywhere p_0, from eq. (23) we arrive at the following forms for the θ, φ, and r components of the surface equation of motion.

$$t_{\theta\theta(1)}n_\theta + t_{\theta r(1)}n_r = (-p_0 + 2H\sigma)\,n_\theta \tag{36}$$

$$t_{\varphi\theta(1)}n_\theta + t_{\varphi r(1)}n_r = 0 \tag{37}$$

$$t_{r\theta(1)}\,n_\theta + t_{rr(1)}n_r = (-p_0 + 2H\sigma)\,n_r \tag{38}$$

The mean curvature H and the physical components of n_i are evaluated from eqs. (33) and (35).

2.3. Applicability of the Solution for the Infinite Cone-Plate Viscometer to the Finite Case

In section 2.1 we discuss the steady-state flow of an incompressible simple fluid of Noll in a cone-plate viscometer. In section 2.2 we derive the boundary conditions at the free surface which the solution of section 2.1 must satisfy if it is to be applicable to the finite cone-plate viscometer.

We find in section 2.1 that there are only four nonzero components of the stress tensor in this case: t_{rr}, $t_{\theta\theta}$, $t_{\varphi\varphi}$, $t_{\varphi\theta}$. From the φ component of the surface equation of motion, eq. (37), we have

$$t_{\varphi\theta(1)}n_\theta = 0 \tag{39}$$

This means that $n_\theta = 0$. From eq. (33) we conclude that $f' = 0$; if eq. (37) is to be satisfied, the free surface must be a surface of constant spherical radius.

Having deduced that $n_\theta = 0$, we have that the θ component of the surface equation of motion, eq. (36), is satisfied identically.

From the r component of the surface equation of motion, eq. (38), we have (we assume that the free surface passes through the outer edge of the cone)

$$t_{rr}\big|_{r=R} = -p_0 + 2H\sigma \tag{40}$$

The mean curvature H may be calculated from eq. (35) to be

$$H = -1/R \tag{41}$$

Equations (2) and (22) yield

$$\begin{aligned} t_{rr}\big|_{r=R} &= t_{\theta\theta}\big|_{r=R} - \sigma_1\big|_{r=R} \\ &= t_{\theta\theta}\big|_{\substack{r=R\\ \theta=\pi/2}} + \rho\psi\big|_{r=R} - \rho\psi\big|_{\substack{r=R\\ \theta=\pi/2}} - \sigma_1\big|_{r=R} - \\ &\qquad \int_{\pi/2}^{\theta} (\sigma_1 - \sigma_2)\big|_{r=R} \cot\theta\, d\theta \end{aligned} \tag{42}$$

If the angle between the cone and the plate is sufficiently small,

$$\begin{aligned} t_{rr}\big|_{r=R} &\doteq t_{\theta\theta}\big|_{\substack{r=R\\ \theta=\pi/2}} - \sigma_1\big|_{\substack{r=R\\ \theta=\pi/2}} - (\sigma_1 - \sigma_2)\big|_{\substack{r=R\\ \theta=\pi/2}} \ln(\sin\theta) \\ &\doteq t_{\theta\theta}\big|_{\substack{r=R\\ \theta=\pi/2}} - \sigma_1\big|_{\substack{r=R\\ \theta=\pi/2}} = t_{rr}\big|_{\substack{r=R\\ \theta=\pi/2}} \end{aligned} \tag{43}$$

Combining eqs. (40) and (43) we finally arrive at a consistent boundary condition for t_{rr}:

$$t_{rr}\Big|_{\substack{r = R \\ \theta = \pi/2}} = -[p_0 + 2\sigma/R] \tag{44}$$

2.4. Discussion

In section 2.3 we conclude that if the analysis of the infinite cone-plate viscometer is to apply to the results from a finite instrument, in principle the free surface must be a surface of constant spherical radius and the angle between the cone and the plate must be small.

At least two factors will influence the shape of the free surface: (*a*) if the instrument is over- or under-filled the shape will be distorted; (*b*) interfacial tension at the cone and at the plate will distort the surface in the neighborhood of these surfaces. Distortion of the free surface from a surface of constant radius would induce secondary flows in the neighborhood of the free surface; such secondary flows have been reported.[13,14] One might guess that the distortion of the free surface would not be important to the experimentalist except in the case where one desired to measure normal stresses near the edge of the cone.

The requirement that the angle between the cone and the plate be small is not new; this is usually done to make inertial effects negligible.[2,3,5]

3. FINITE PARALLEL-PLATE VISCOMETER

Our discussion of this instrument will parallel the discussion in section 2 for the cone-plate viscometer, and hence will be somewhat abbreviated. The procedure is to ask whether the solution for the infinite geometry satisfies the necessary boundary conditions at the free surface.

3.1. Infinite Parallel-Plate Viscometer

The problem is best described in a cylindrical coordinate system, the z-axis of which is the axis of revolution. The z component of torque required to drive the upper plate (located at $z = L$) at a constant angular velocity Ω is $-T_z$; the lower plate (located at $z = 0$) is fixed in space.

Following the discussion by Coleman and Noll for the class of viscometric flows of simple fluids,[7,8] we assume that there is only one nonzero component of velocity, v_θ, and that (v_θ/r) is a function of z alone. Under these conditions there are four nonzero components of stress ($t_{\theta z}$, t_{rr}, $t_{\theta\theta}$, t_{zz}) and the behavior of a simple fluid of Noll in this geometry is described by the same three material functions introduced in eqs. (1)–(3), but expressed in terms of the variables appropriate to this problem.

$$\tau(\kappa) = t_{\theta z} \tag{45}$$

$$\sigma_1(\kappa) = t_{zz} - t_{rr} \tag{46}$$

$$\sigma_2(\kappa) = t_{\theta\theta} - t_{rr} \tag{47}$$

$$\kappa = r \frac{d}{dz} (v_\theta/r) \tag{48}$$

As with the cone-plate viscometer, if there is to be only one nonzero component of velocity, inertial effects must be neglected[2,3] and the equations of motion become

$$0 = -\frac{\partial \Phi}{\partial r} + \frac{\partial \tau_{rr}}{\partial r} - \frac{\sigma_2}{r} \tag{49}$$

$$0 = -\frac{\partial \Phi}{\partial \theta} + \frac{\partial (r\,\tau_{\theta z})}{\partial z} \tag{50}$$

$$0 = -\frac{\partial \Phi}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} \tag{51}$$

Since Φ must be a periodic function of θ, from eq. (50) $\partial\Phi/\partial\theta = 0$ and

$$r\,\tau_{\theta z} = \text{constant} = C_2 \tag{52}$$

The rectangular Cartesian components of the torque vector T_m may be calculated from eq. (13); in particular we find

$$T_z = \pi C_2 R^2 \tag{53}$$

Elimination of C_2 between eqs. (52) and (53) yields

$$\tau_{\theta z} = t_{\theta z} = T_z/\pi\, R^2 r = \tau(\varkappa) \tag{54}$$

If τ is an invertible function

$$\varkappa = r\,\frac{d}{dz}\,(v_\theta/r) = \tau^{-1}(T_z/\pi\, R^2 r) \tag{55}$$

Since $\tau^{-1}(T_z/\pi\, R^2 r)$ is not a function of z, this may be integrated to yield

$$\Omega = \int_0^L \frac{d}{dz}\,(v_\theta/r)\, dz = (L/r)\,\tau^{-1}\,(T_z/\pi\, R^2 r) \tag{56}$$

By eq. (55) $\varkappa$ is not a function of z and $\partial\tau_{zz}/\partial z$ in eq. (51) is identically zero. From the r and z components of the equation of motion we have

$$\Phi - \Phi\Big|_{\substack{r = R \\ z = 0}} = \int_R^r \left[\frac{d\tau_{rr}}{dr} - \frac{\sigma_2}{r}\right] dr \tag{57}$$

which upon rearrangement yields

$$t_{rr} = t_{rr}\Big|_{\substack{r = R \\ z = 0}} + \int_R^r \frac{\sigma_2}{r}\, dr + \rho\psi - \rho\psi\Big|_{\substack{r = R \\ z = 0}} \tag{58}$$

3.2. Boundary Conditions at the Free Surface in the Finite Parallel-Plate Viscometer

Equations (23)–(28) again apply, but we choose the surface coordinates to be

$$\begin{aligned} u^1 &= x^1 = \theta \\ u^2 &= x^2 = z \end{aligned} \tag{59}$$

On the surface

$$x^3 = r = f(u^2) \tag{60}$$

From eq. (27) the components of the surface metric tensor are computed to be

$$\begin{aligned} a_{11} &= (f)^2 \\ a_{12} &= 0 \\ a_{22} &= 1 + (f')^2 \end{aligned} \tag{61}$$

We denote the liquid in the viscometer as phase 1 and the surrounding air as phase 2 so that eq. (32) again applies. By eq. (24)

$$\begin{aligned} n_1 &= rn_\theta = 0 \\ n_2 &= n_z = \frac{-f'}{[1 + (f')^2]^{1/2}} \\ n_3 &= n_r = \frac{1}{[1 + (f')^2]^{1/2}} \end{aligned} \tag{62}$$

Finally eqs. (28) and (26) yield

$$\begin{aligned} b_{11} &= \frac{-f}{[1 + (f')^2]^{1/2}} \\ b_{12} &= 0 \\ b_{22} &= \frac{f''}{[1 + (f')^2]^{1/2}} \end{aligned} \tag{63}$$

$$H = \frac{ff'' - (f')^2 - 1}{2f[1 + (f')^2]^{3/2}} \tag{64}$$

Neglecting all viscous forces in the surrounding air stream where the pressure is everywhere p_0, from eq. (23) we arrive at the following forms for the θ, z, and r components of the surface equation of motion.

$$t_{\theta z(1)}\, n_z + t_{\theta r(1)}\, n_r = 0 \tag{65}$$

$$t_{zz(1)}\, n_z + t_{zr(1)}\, n_r = (-p_0 + 2H\sigma)n_z \tag{66}$$

$$t_{rz(1)}\, n_z + t_{rr(1)}\, n_r = (-p_0 + 2H\sigma)n_r \tag{67}$$

Equations (62) and (64) yield the mean curvature H and the physical components of n_i.

3.3. Applicability of the Solution for the Infinite Parallel-Plate Viscometer to the Finite Case

We now wish to see whether the solution for the infinite geometry discussed in section 3.1 does indeed satisfy the boundary conditions at the free surface in the finite viscometer.

In section 3.1 we found that there were only four nonzero components of the stress tensor: $t_{\theta z}$, t_{rr}, $t_{\theta\theta}$, t_{zz}. From the θ component of the surface equation of motion, eq. (65)

$$t_{\theta z(1)}\, n_z = 0 \tag{68}$$

which implies that $n_z = 0$ and $f' = 0$. This means that, if eq. (65) is to be satisfied, the free surface must be a surface of constant radius in cylindrical coordinates.

Since $n_z = 0$ and $t_{zr} = 0$, we have that the z component of the surface equation, eq. (66), is satisfied identically.

The r component of the surface equation of motion, eq. (67) reduces to

$$t_{rr(1)} = t_{rr}\Big|_{r=R} = -\left(p_0 + \frac{\sigma}{R}\right) \tag{69}$$

Since changes in potential energy will be small in any practical situation, from eq. (58)

$$t_{rr}\Big|_{r=R} \doteq t_{rr}\Big|_{\substack{r=R\\ z=0}} \tag{70}$$

which when inserted into eq. (69) yields a consistent boundary condition for t_{rr}:

$$t_{rr}\Big|_{\substack{r=R\\ z=0}} \doteq -\left(p_0 + \frac{\sigma}{R}\right) \tag{71}$$

3.4. Discussion

If the analysis of the infinite parallel-plate viscometer in section 3.1 is to apply to the results from a finite instrument, we find in section 3.3 that the free surface must be a surface of constant radius in cylindrical coordinates. Factors influencing the shape of the free surface are discussed in section 2.4. Probably the shape of the free surface is not too important to the experimentalist unless he attempts to measure normal stresses near the edge of the plate.

APPENDIX

Notation

$a_{\alpha\beta}$	Surface metric tensor;[11d] defined by eq. (27).
$b_{\alpha\beta}$	Symmetric tensor in Gauss's formulae.[11c]
C_1	Defined by eq. (11).
C_2	Defined by eq. (52).
f	In section 2.2, defined by eq. (30); in section 3.2, defined by eq. (60).
f', f''	Derivatives of f.
f_i	External body force vector per unit mass.

g_{ij} Metric tensor.[11e]

H Mean curvature of the surface;[11c] defined by eq. (27).

n_i In section 2.1, this is the unit normal vector to a surface directed into the fluid in question; in sections 2.2 and 3.2, this is the unit vector normal to the surface such that ($x^i_{;1}$, $x^i_{;2}$, n^i) have the same orientation as the tangents to the spatial coordinate curves;[11b] defined by eq. (24).

$n^j_{(K)}$ Unit vector normal to the surface and outwardly directed into phase K.

n_r, n_θ, n_φ Physical components[11a] of n_i in section 2.2.

n_r, n_θ, n_z Physical components[11a] of n_i in section 3.2.

p Pressure.

p_0 Ambient pressure of air at the free surface.

r In section 2, the radius in spherical coordinates; in section 3, the radius in cylindrical coordinates.

R Radius of cone in spherical coordinates in section 2; radius of plate in cylindrical coordinates in section 3.

R^j Position vector; defined as $(y^i/\partial x^j/\partial y^i)$.

t^k Stress vector. This is the force per unit area which a fluid exerts at a point on the surface where the unit normal directed into the fluid is n_i; related to the stress tensor by $t^k = t^{ki} n_i$.

t_{ij} Stress tensor.

t_{rr}, $t_{\theta\theta}$, $t_{\varphi\varphi}$, $t_{\theta\varphi}$, $t_{\theta r}$, $t_{r\varphi}$ Physical components of the stress tensor t_{ij} in spherical coordinates; used in section 2.

t_{rr}, $t_{\theta\theta}$, t_{zz}, t_{rz}, $t_{r\theta}$, $t_{\theta z}$ Physical components of the stress tensor t_{ij} in cylindrical coordinates; used in section 3.

$t^{ij}_{(K)}$ Stress tensor adjacent to the surface in phase K.

T_z z-Component of the torque vector T_i in rectangular Cartesian components.

T_i In section 2, torque vector which the fluid exerts on the cone; in section 3, torque vector which the fluid exerts on the upper plate.

u^α Surface coordinates; defined in section 2.2 by eqs. (29), defined in section 3.2 by eqs. (59).

v_r, v_θ, v_φ Physical components of the velocity vector in spherical coordinates; used in section 2.

v_r, v_θ, v_z Physical components of the velocity vector in cylindrical coordinates; used in section 3.

x^i Curvilinear coordinates.

$x^i_{;\alpha}$ Defined by eq. (25).

y^i Rectangular Cartesian coordinates.

ϵ_{ijk} Defined as $\sqrt{g}\, e_{ijk}$, where g is the determinant of the matrix of g_{mn} and e_{ijk} is the skew-symmetric relative tensor such that[11f] $e_{123} = 1$.

θ In section 2, the spherical coordinate measured from the positive z axis; in section 3, the cylindrical coordinate measured from the positive x axis.

Θ Value of θ at the cone in section 2.

$\varkappa$ Defined for the cone-plate viscometer by eq. (4) and for the parallel-plate viscometer by eq. (48).

ρ Density.

σ Surface tension.

$\sigma_1(\varkappa)$, $\sigma_2(\varkappa)$, $\tau(\varkappa)$ Material functions which describe the behavior of a simple incompressible fluid of Noll in the class of viscometric flows. Defined for the cone-plate viscometer by eqs. (1)–(3) and for the parallel-plate viscometer by eqs. (45)–(47).

τ_{rr}, $\tau_{\theta\theta}$, $\tau_{\varphi\varphi}$, $\tau_{\theta\varphi}$, $\tau_{\theta r}$, $\tau_{r\varphi}$ Physical components of the viscous portion of the stress tensor τ_{ij} in spherical coordinates; used in section 2.

τ_{rr}, $\tau_{\theta\theta}$, τ_{zz}, τ_{rz}, $\tau_{r\theta}$, $\tau_{\theta z}$ Physical components of the viscous portion of the stress tensor τ_{ij} in cylindrical coordinates; used in section 3.

τ_{ij} Viscous portion of the stress tensor t_{ij}; defined as $[t_{ij} + p\, g_{ij}]$.

φ In section 2, the spherical coordinate measured in the x–y plane from the positive x axis.

Φ Defined by eq. (7).

ψ External body force potential defined by eq. (6).

Ω Angular velocity of the cone in section 2; angular velocity of the upper plate in section 3.1.

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Résumé

Les analyses de l'écoulement d'un fluide simple de Noll, à la fois dans un viscosimètre à plateau conique infini et un viscosimètre à plateaux parallèles infinis sont compatibles avec l'existence d'une surface libre dans les instruments limités. On a négligé dans la description des surfaces libres: l'effet de tout gradient dans la tension superficielle, l'effet de la vitesse de déformation de la surface libre par suite de la force de tension de la surface, l'effet du transfert de masse à la surface libre, les effets dûs à l'inertie et à la pesanteur sur la surface libre, et la distorsion de la surface libre contre les parois du solide.

Zusammenfassung

Die Fliessanalyse einer einfachen Flüssigkeit nach Noll ist sowohl in einem randeffektfreien Kegelplattenviskosimeter als auch in einem randeffektfreien Parallelplattenviskosimeter konsistent mit der Existenz einer freien Oberfläche bei den Instrumenten mit Randeffekt. In der Beschreibung der freien Oberfläche werden die Einflüsse des Oberflächenspannungsgradienten, der Einfluss der Deformationsgeschwindigkeit der freien Oberfläche auf den Spannungstensor, der Einfluss der Massenübertragung an der freien Oberfläche, Trägheits- und Gravitationseinflüsse in der freien Oberfläche sowie Verzerrung der freien Oberfläche an festen Grenzen vernachlässigt.

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